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Position and Velocity Analysis of Spatial Cam Mechanisms

M. Kemal Özgören
Middle East Technical University
Ankara, Turkey

Abstract— *This paper presents the position and velocity analysis of a spatial cam mechanism, which consists of two cams that are connected to the base with cylindrical joints. By fixing two of the four parameters of the cylindrical joints, the cams can be operated in either of the oscillating and reciprocating modes. The analysis includes writing the loop closure and cam contact equations both in the position and velocity levels and then solving them for the unspecified variables and their rates in correspondence with the specified ones. The solution in the position level has been obtained in a semi-analytical way by reducing the number of equations that necessitate a numerical solution from eleven to two. On the other hand, it has been possible to obtain the solution in the velocity level analytically owing to the linearity of the velocity equations. Writing and solving the equations have been facilitated by introducing the contact ratio between the gradient vectors as an auxiliary variable.*

Keywords: Spatial Mechanisms, Cam Mechanisms, Surface Contact, Contact Ratio, Higher Kinematic Pairs, Rolling Kinematic Pairs

I. Introduction

There are numerous previous publications about the spatial cam mechanisms. Some of the typical ones can be seen in the list of references. In [1] and [2], Dandhe and Chakraborty present the basics of curvature analysis. In [7] and [8], Yan and Cheng also present curvature analysis with more specific applications. In [3], Angeles and Lopez-Cajun discuss the optimization of cylindrical cams. In [4], Gonzales-Palacios and Angeles present synthesis examples of mechanisms consisting of spherical cams and oscillating roller followers. In [5], Tsai and Wei present a method based on the envelope theory for the determination of planar and spatial cam profiles. In [9], Yang also presents a similar method for the determination of spherical cam profiles. In [6], Ramahi and Tokad present a kinematic analysis based on the screw theory oriented to the generation of contact surfaces for the three-link spatial, spherical and planar cam mechanisms. In [10], Kim, Sacks, and Joskowicz present a kinematic analysis method based on configuration spaces, by means of which a mechanical part with a complex shape can be modeled as a combination of several patches with simple shapes. The same authors mention that their method can also handle the problem of contact changes. In [11], Cheng, Jiang, and Wang present a method of modeling spatial cams based on an analysis of conjugate surfaces by using the principles of differential geometry. In [12], Ozgoren presents the position analysis of a spatial cam mechanism as one of the examples. That mechanism consists of an ellipsoidal cam and a cylindrical cam connected to the base with revolute joints. In [12], as an originality in the kinematic description of the cam pairs, the *contact ratio* is introduced between the gradient vectors of the cam surfaces as an auxiliary variable. As demonstrated in [12] and also in this paper, the contact ratio facilitates expressing the kinematic relationships between two cams.

In this paper, as compared to the example in [12], a somewhat different three-link cam mechanism is considered. It consists of two cams, which are connected to the base with cylindrical joints. Thus, by fixing two of the four variables of the cylindrical joints, it becomes possible to use the mechanism in one of the four different operational modes, which are rotation-rotation, rotation-translation, translation-rotation, and translation-translation. Both cams are assumed to be shaped as smooth surfaces without sharp edges and corners. The kinematic analysis of this mechanism is carried out both in the position and velocity levels, whereas the kinematic analysis in [12] was carried out only in the position level. Other than this difference, the notation and the methodology used in this paper is the same as those used in [12]. The kinematic analysis includes writing the loop closure and cam contact equations, differentiating them to obtain the velocity equations, and then solving all those equations to find the the unspecified variables and their rates that correspond to the specified ones. For the considered mechanism, the solution in the position level can only be obtained in a semi-analytical way. That is, by means of symbolic manipulations, the number of equations that necessitate a numerical solution can be reduced from eleven to two. On the other hand, the solution in the velocity level can be obtained analytically owing to the linearity of the velocity equations.

II. Kinematic Description of a Mechanical System

This section is devoted to explain the notation and terminology that is used in this paper in order to describe the kinematic relationships among the links of a general mechanical system established by the joints, i.e., the kinematic pairs.

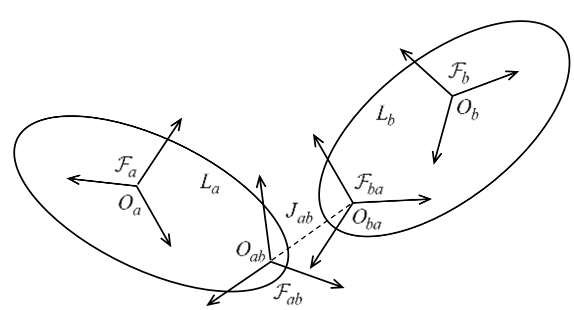

Figure 1. Two Links Connected by a Joint

A. Description of the Link and Joint Frames

Consider a spatial mechanical system that consists of several links such as L_a, L_b, L_c, etc. Two connected links of such a system are depicted in Figure 1. The links are assumed to be rigid. The joint between L_a and L_b is denoted as J_{ab} or J_{ba}. On each link, e.g., L_a, there are attached several reference frames such as $\mathcal{F}_a$, $\mathcal{F}_{ab}$, $\mathcal{F}_{ac}$, $\mathcal{F}_{ad}$, etc. Here, $\mathcal{F}_a$ is the *link frame* and the others ($\mathcal{F}_{ab}$, $\mathcal{F}_{ac}$, $\mathcal{F}_{ad}$, etc) are the *joint frames*. The joint frame $\mathcal{F}_{ab}$ is attached to the kinematic element E_{ab} of J_{ab} on L_a. The origins of the frames $\mathcal{F}_a$ and $\mathcal{F}_{ab}$ are O_a and O_{ab}. Their *k*-th unit basis vectors are $\vec{u}_k^{(a)}$ and $\vec{u}_k^{(ab)}$.

B. Kinematic Connectivity Equations

Considering two links L_a and L_b connected by a joint J_{ab}, such as those shown in Figure 1, the relative position (orientation and origin location) of the link frame $\mathcal{F}_b$ with respect to the link frame $\mathcal{F}_a$ can be expressed as explained below.

a) *Orientation Equation*

The overall orientation of $\mathcal{F}_b$ with respect to $\mathcal{F}_a$ can be expressed as follows by taking the companion joint frames $\mathcal{F}_{ab}$ and $\mathcal{F}_{ba}$ also into account.

$$\hat{C}^{(a,b)} = \hat{C}^{(a,ab)}\hat{C}^{(ab,ba)}\hat{C}^{(ba,b)} \tag{2.1}$$

In Eq. (2.1), $\hat{C}^{(a,ab)}$ and $\hat{C}^{(ba,b)}$ are constant matrices but $\hat{C}^{(ab,ba)}$ is a variable matrix, which is a function of the variable or variables of the joint J_{ab}.

b) *Location Equation*

The vector equation that expresses the location of the origin O_b with respect to the origin O_a is

$$\vec{r}_{a,b} = \vec{r}_{a,ab} + \vec{r}_{ab,ba} + \vec{r}_{ba,b} \tag{2.2}$$

In Eq. (2.2), as a generic symbol, $\vec{r}_{p,q}$ denotes the relative location vector directed from an origin O_p to another origin O_q. As expressed in $\mathcal{F}_a$, the matrix equation corresponding to Eq. (2.2) is written as

$$\bar{r}_{a,b}^{(a)} = \bar{r}_{a,ab}^{(a)} + \bar{r}_{ab,ba}^{(a)} + \bar{r}_{ba,b}^{(a)} \tag{2.3}$$

Note that $\vec{r}_{a,ab}$ and $\vec{r}_{ba,b}$ appear constant in $\mathcal{F}_a$ and $\mathcal{F}_b$, respectively. Moreover, $\vec{r}_{ab,ba}$ has the simplest expression in $\mathcal{F}_{ab}$. Therefore, Eq. (2.3) can be written more expressively as

$$\bar{r}_{a,b}^{(a)} = \bar{r}_{a,ab}^{(a)} + \hat{C}^{(a,ab)}\bar{r}_{ab,ba}^{(ab)} + \hat{C}^{(a,b)}\bar{r}_{ba,b}^{(b)} \tag{2.4}$$

Upon substituting Eq. (2.1) into Eq. (2.4), a more detailed equation is obtained as

$$\bar{r}_{a,b}^{(a)} = \bar{r}_{a,ab}^{(a)} + \hat{C}^{(a,ab)}[\bar{r}_{ab,ba}^{(ab)} + \hat{C}^{(ab,ba)}\hat{C}^{(ba,b)}\bar{r}_{ba,b}^{(b)}] \tag{2.5}$$

In Eq'ns (2.4) and (2.5), $\bar{r}_{a,ab}^{(a)}$ and $\bar{r}_{ba,b}^{(b)}$ are constant column matrices but $\bar{r}_{ab,ba}^{(ab)}$ is a variable column matrix, which is a function of the variable or variables of the joint J_{ab}.

III. Kinematic Description of the Analyzed Three-Link Spatial Cam Mechanism

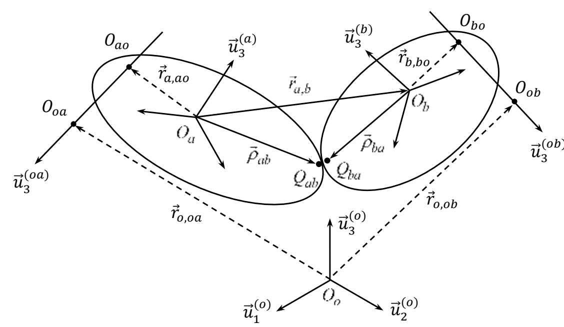

Figure 2. The Analyzed Three-Link Spatial Cam Mechanism

The spatial cam mechanism considered in this paper is illustrated in Figure 2. It is a three-link mechanism that consists of two cams (L_a and L_b), which are connected to the base link (L_o) with cylindrical joints. Thus, by fixing two of the four variables of the cylindrical joints, it becomes possible to use the mechanism in one of the four different operational modes, which are rotation-rotation, rotation-translation, translation-rotation, and translation-translation. Both cams are assumed to be shaped as smooth surfaces without sharp edges and corners. The cams themselves are their own kinematic elements of the cam joint $J_{ab} = J_{ba}$ between them. Therefore, for the sake of convenience, $\mathcal{F}_a$ and $\mathcal{F}_b$ are selected so that

$$\mathcal{F}_a = \mathcal{F}_{ab} \quad \text{and} \quad \mathcal{F}_b = \mathcal{F}_{ba}$$

The axes of the cylindrical joints $J_{oa} = J_{ao}$ and $J_{ob} = J_{bo}$ are represented by the unit vectors $\vec{u}_3^{(oa)} = \vec{u}_3^{(ao)}$ and $\vec{u}_3^{(ob)} = \vec{u}_3^{(bo)}$. The origins of the joint frames $\mathcal{F}_{oa}$, $\mathcal{F}_{ao}$, $\mathcal{F}_{ob}$, and $\mathcal{F}_{bo}$ are O_{oa}, O_{ao}, O_{ob}, and O_{bo}, respectively. O_{oa} and O_{ob} are the orthogonal projections of O_o on the axis of $J_{oa} = J_{ao}$ and $J_{ob} = J_{bo}$. Similarly, O_{ao} and O_{bo} are the orthogonal projections of O_a and O_b on the axes of $J_{oa} = J_{ao}$ and $J_{ob} = J_{bo}$, respectively.

IV. Loop Closure and Cam Contact Equations

A. Loop Closure Equation for Link Orientations

$$\hat{C}^{(o,a)}\hat{C}^{(a,b)} = \hat{C}^{(o,b)} \;\Rightarrow$$

$$\hat{C}^{(o,oa)}\hat{C}^{(oa,ao)}\hat{C}^{(ao,a)}\hat{C}^{(a,b)} = \hat{C}^{(o,ob)}\hat{C}^{(ob,bo)}\hat{C}^{(bo,b)} \tag{4.1}$$

The matrices in Eq. (4.1) can be expressed separately as follows.

$$\hat{C}^{(o,oa)} = \hat{R}_3(\alpha_{oa})\hat{R}_2(\beta_{oa})\hat{R}_1(\gamma_{oa}) \tag{4.2}$$

$$\hat{C}^{(o,ob)} = \hat{R}_3(\alpha_{ob})\hat{R}_2(\beta_{ob})\hat{R}_1(\gamma_{ob}) \tag{4.3}$$

$$\hat{C}^{(a,ao)} = \hat{R}_3(\alpha_{ao})\hat{R}_2(\beta_{ao})\hat{R}_1(\gamma_{ao}) \tag{4.4}$$

$$\hat{C}^{(b,bo)} = \hat{R}_3(\alpha_{bo})\hat{R}_2(\beta_{bo})\hat{R}_1(\gamma_{bo}) \tag{4.5}$$

$$\hat{C}^{(oa,ao)} = \hat{R}_3(\theta_{oa}) \tag{4.6}$$

$$\hat{C}^{(ob,bo)} = \hat{R}_3(\theta_{ob}) \tag{4.7}$$

$$\hat{C}^{(a,b)} = \hat{R}_3(\phi_{ab})\hat{R}_2(\theta_{ab})\hat{R}_3(\psi_{ab}) \tag{4.8}$$

Here, the rotation sequence of $\hat{C}^{(a,b)}$ is selected as 3-2-3 for the sake of reciprocal symmetry so that the following alternative closure equation looks similar.

$$\hat{C}^{(o,b)}\hat{C}^{(b,a)} = \hat{C}^{(o,a)}$$

In the preceding equations, $\hat{R}_k(\theta)$ is the *k*-th basic rotation matrix, which is defined as follows according to the Rodrigues formula.

$$\hat{R}_k(\theta) = e^{\tilde{u}_k\theta} = \hat{I}\cos\theta + \tilde{u}_k\sin\theta + \bar{u}_k\bar{u}_k^t(1-\cos\theta) \tag{4.9}$$

In Eq. (4.9), $\tilde{u}_k$ is the skew symmetric matrix generated from the basic column matrix $\bar{u}_k$. The basic column matrices are

$$\bar{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\;\; \bar{u}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix},\;\; \bar{u}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \tag{4.10}$$

For an arbitrary column matrix $\bar{c}$, the corresponding skew symmetric matrix $\tilde{c}$ is generated by the ssm operator, which is defined as described below.

$$\bar{c} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \;\Rightarrow\; \tilde{c} = \mathrm{ssm}(\bar{c}) = \begin{bmatrix}0 & -c_3 & c_2\\ c_3 & 0 & -c_1\\ -c_2 & c_1 & 0\end{bmatrix} \tag{4.11}$$

Upon the due substitutions, Eq. (4.1) becomes

$$\begin{aligned}&(e^{\tilde{u}_3\alpha_{oa}}e^{\tilde{u}_2\beta_{oa}}e^{\tilde{u}_1\gamma_{oa}})(e^{\tilde{u}_3\theta_{oa}})(e^{-\tilde{u}_1\gamma_{ao}}e^{-\tilde{u}_2\beta_{ao}}e^{-\tilde{u}_3\alpha_{ao}})\times\\&(e^{\tilde{u}_3\phi_{ab}}e^{\tilde{u}_2\theta_{ab}}e^{\tilde{u}_3\psi_{ab}})\\&=(e^{\tilde{u}_3\alpha_{ob}}e^{\tilde{u}_2\beta_{ob}}e^{\tilde{u}_1\gamma_{ob}})(e^{\tilde{u}_3\theta_{ob}})(e^{-\tilde{u}_1\gamma_{bo}}e^{-\tilde{u}_2\beta_{bo}}e^{-\tilde{u}_3\alpha_{bo}})\end{aligned} \tag{4.12}$$

In Eq. (4.12), the constant matrix products can be denoted as follows for the sake of notational brevity.

$$\hat{M}_{oa} = e^{\tilde{u}_3\alpha_{oa}}e^{\tilde{u}_2\beta_{oa}}e^{\tilde{u}_1\gamma_{oa}} \tag{4.13}$$

$$\hat{M}_{ao} = e^{\tilde{u}_3\alpha_{ao}}e^{\tilde{u}_2\beta_{ao}}e^{\tilde{u}_1\gamma_{ao}} \tag{4.14}$$

$$\hat{M}_{ob} = e^{\tilde{u}_3\alpha_{ob}}e^{\tilde{u}_2\beta_{ob}}e^{\tilde{u}_1\gamma_{ob}} \tag{4.15}$$

$$\hat{M}_{bo} = e^{\tilde{u}_3\alpha_{bo}}e^{\tilde{u}_2\beta_{bo}}e^{\tilde{u}_1\gamma_{bo}} \tag{4.16}$$

Then, Eq. (4.12) can be written briefly as

$$\hat{M}_{oa}e^{\tilde{u}_3\theta_{oa}}\hat{M}_{ao}^t e^{\tilde{u}_3\phi_{ab}}e^{\tilde{u}_2\theta_{ab}}e^{\tilde{u}_3\psi_{ab}} = \hat{M}_{ob}e^{\tilde{u}_3\theta_{ob}}\hat{M}_{bo}^t \tag{4.17}$$

B. Loop Closure Equation for Origin Locations

$$\begin{aligned}&\overrightarrow{O_oO_{oa}} + \overrightarrow{O_{oa}O_{ao}} + \overrightarrow{O_{ao}O_a} + \overrightarrow{O_aQ_{ab}} \;\Rightarrow\\&= \overrightarrow{O_oO_{ob}} + \overrightarrow{O_{ob}O_{bo}} + \overrightarrow{O_{bo}O_b} + \overrightarrow{O_bQ_{ba}}\end{aligned}$$

$$\begin{aligned}&\vec{r}_{o,oa} + s_{oa}\vec{u}_3^{(oa)} - \vec{r}_{a,ao} + \vec{\rho}_{ab}\\&= \vec{r}_{o,ob} + s_{ob}\vec{u}_3^{(oa)} - \vec{r}_{b,bo} + \vec{\rho}_{ba}\end{aligned} \tag{4.18}$$

The above-written vector equation leads to the following matrix equation as expressed in $\mathcal{F}_o$.

$$\begin{aligned}&\bar{r}_{o,oa}^{(o)} + s_{oa}\bar{u}_3^{(oa/o)} - \bar{r}_{a,ao}^{(o)} + \bar{\rho}_{ab}^{(o)}\\&= \bar{r}_{o,ob}^{(o)} + s_{ob}\bar{u}_3^{(ob/o)} - \bar{r}_{b,bo}^{(o)} + \bar{\rho}_{ba}^{(o)}\end{aligned} \tag{4.19}$$

Considering the natural resolution frames for the relevant vectors, in which they have the simplest expressions, the following equations can be written.

$$\bar{r}_{o,oa}^{(o)} = r_{ooa}\bar{u}_1^{(oa/o)} = r_{ooa}\hat{C}^{(o,oa)}\bar{u}_1 \;\Rightarrow$$

$$\bar{r}_{o,oa}^{(o)} = r_{ooa}e^{\tilde{u}_3\alpha_{oa}}e^{\tilde{u}_2\beta_{oa}}e^{\tilde{u}_1\gamma_{oa}}\bar{u}_1 = r_{ooa}e^{\tilde{u}_3\alpha_{oa}}e^{\tilde{u}_2\beta_{oa}}\bar{u}_1 \tag{4.20}$$

$$\bar{r}_{o,ob}^{(o)} = r_{oob}\bar{u}_1^{(ob/o)} = r_{oob}\hat{C}^{(o,ob)}\bar{u}_1 \;\Rightarrow$$

$$\bar{r}_{o,ob}^{(o)} = r_{oob}e^{\tilde{u}_3\alpha_{ob}}e^{\tilde{u}_2\beta_{ob}}e^{\tilde{u}_1\gamma_{ob}}\bar{u}_1 = r_{oob}e^{\tilde{u}_3\alpha_{ob}}e^{\tilde{u}_2\beta_{ob}}\bar{u}_1 \tag{4.21}$$

$$\bar{u}_3^{(oa/o)} = \hat{C}^{(o,oa)}\bar{u}_3^{(oa/oa)} = e^{\tilde{u}_3\alpha_{oa}}e^{\tilde{u}_2\beta_{oa}}e^{\tilde{u}_1\gamma_{oa}}\bar{u}_3 \tag{4.22}$$

$$\bar{u}_3^{(ob/o)} = \hat{C}^{(o,ob)}\bar{u}_3^{(ob/ob)} = e^{\tilde{u}_3\alpha_{ob}}e^{\tilde{u}_2\beta_{ob}}e^{\tilde{u}_1\gamma_{ob}}\bar{u}_3 \tag{4.23}$$

$$\begin{aligned}\bar{\rho}_{ab}^{(o)} &= \hat{C}^{(o,a)}\bar{\rho}_{ab}^{(a)} = \hat{C}^{(o,a)}\bar{\rho}_a\\&= \hat{C}^{(o,a)}(\bar{u}_1x_a + \bar{u}_2y_a + \bar{u}_3z_a)\end{aligned} \tag{4.24}$$

$$\begin{aligned}\bar{\rho}_{ba}^{(o)} &= \hat{C}^{(o,b)}\bar{\rho}_{ba}^{(b)} = \hat{C}^{(o,b)}\bar{\rho}_b\\&= \hat{C}^{(o,b)}(\bar{u}_1x_b + \bar{u}_2y_b + \bar{u}_3z_b)\end{aligned} \tag{4.25}$$

$$\bar{r}_{a,ao}^{(o)} = \hat{C}^{(o,a)}\bar{r}_{a,ao}^{(a)} = \hat{C}^{(o,a)}(r_{aao}e^{\tilde{u}_3\alpha_{ao}}e^{\tilde{u}_2\beta_{ao}}\bar{u}_1) \;\Rightarrow$$

$$\begin{aligned}\bar{r}_{a,ao}^{(o)} &= (e^{\tilde{u}_3\alpha_{oa}}e^{\tilde{u}_2\beta_{oa}}e^{\tilde{u}_1\gamma_{oa}})(e^{\tilde{u}_3\theta_{oa}})(e^{-\tilde{u}_1\gamma_{ao}}e^{-\tilde{u}_2\beta_{ao}}e^{-\tilde{u}_3\alpha_{ao}})\times\\&\times(r_{aao}e^{\tilde{u}_3\alpha_{ao}}e^{\tilde{u}_2\beta_{ao}}\bar{u}_1) \;\Rightarrow\end{aligned}$$

$$\bar{r}_{a,ao}^{(o)} = r_{aao}e^{\tilde{u}_3\alpha_{oa}}e^{\tilde{u}_2\beta_{oa}}e^{\tilde{u}_1\gamma_{oa}}e^{\tilde{u}_3\theta_{oa}}\bar{u}_1 \tag{4.26}$$

Similarly,

$$\bar{r}_{b,bo}^{(o)} = r_{bbo} e^{\tilde{u}_3\alpha_{ob}} e^{\tilde{u}_2\beta_{ob}} e^{\tilde{u}_1\gamma_{ob}} e^{\tilde{u}_3\theta_{ob}} \bar{u}_1 \tag{4.27}$$

Again, upon the due substitutions, Eq. (4.19) becomes

$$\begin{aligned}
& r_{ooa} e^{\tilde{u}_3\alpha_{oa}} e^{\tilde{u}_2\beta_{oa}} \bar{u}_1 + s_{oa} e^{\tilde{u}_3\alpha_{oa}} e^{\tilde{u}_2\beta_{oa}} e^{\tilde{u}_1\gamma_{oa}} \bar{u}_3 \\
& - r_{aao} e^{\tilde{u}_3\alpha_{oa}} e^{\tilde{u}_2\beta_{oa}} e^{\tilde{u}_1\gamma_{oa}} e^{\tilde{u}_3\theta_{oa}} \bar{u}_1 \\
& + e^{\tilde{u}_3\alpha_{oa}} e^{\tilde{u}_2\beta_{oa}} e^{\tilde{u}_1\gamma_{oa}} e^{\tilde{u}_3\theta_{oa}} e^{-\tilde{u}_1\gamma_{ao}} e^{-\tilde{u}_2\beta_{ao}} e^{-\tilde{u}_3\alpha_{ao}} \times \\
& \times (\bar{u}_1 x_a + \bar{u}_2 y_a + \bar{u}_3 z_a) \\
& = r_{oob} e^{\tilde{u}_3\alpha_{ob}} e^{\tilde{u}_2\beta_{ob}} \bar{u}_1 + s_{ob} e^{\tilde{u}_3\alpha_{ob}} e^{\tilde{u}_2\beta_{ob}} e^{\tilde{u}_1\gamma_{ob}} \bar{u}_3 \\
& - r_{bbo} e^{\tilde{u}_3\alpha_{ob}} e^{\tilde{u}_2\beta_{ob}} e^{\tilde{u}_1\gamma_{ob}} e^{\tilde{u}_3\theta_{ob}} \bar{u}_1 \\
& + e^{\tilde{u}_3\alpha_{ob}} e^{\tilde{u}_2\beta_{ob}} e^{\tilde{u}_1\gamma_{ob}} e^{\tilde{u}_3\theta_{ob}} e^{-\tilde{u}_1\gamma_{bo}} e^{-\tilde{u}_2\beta_{bo}} e^{-\tilde{u}_3\alpha_{bo}} \times \\
& \times (\bar{u}_1 x_b + \bar{u}_2 y_b + \bar{u}_3 z_b)
\end{aligned} \tag{4.28}$$

Eq. (4.28) can also be written briefly as follows with the help of Eq'ns (4.13) to (4.16).

$$\begin{aligned}
& \hat{M}_{oa}(r_{ooa}\bar{u}_1 + s_{oa}\bar{u}_3 - r_{aao} e^{\tilde{u}_3\theta_{oa}} \bar{u}_1) \\
& + \hat{M}_{oa} e^{\tilde{u}_3\theta_{oa}} \hat{M}_{ao}^t (\bar{u}_1 x_a + \bar{u}_2 y_a + \bar{u}_3 z_a) \\
& = \hat{M}_{ob}(r_{oob}\bar{u}_1 + s_{ob}\bar{u}_3 - r_{bbo} e^{\tilde{u}_3\theta_{ob}} \bar{u}_1) \\
& + \hat{M}_{ob} e^{\tilde{u}_3\theta_{ob}} \hat{M}_{bo}^t (\bar{u}_1 x_b + \bar{u}_2 y_b + \bar{u}_3 z_b)
\end{aligned} \tag{4.29}$$

Eq. (4.29) can be manipulated further to

$$\begin{aligned}
& \hat{M}_{oa}[\bar{u}_1(r_{ooa} - r_{aao}\cos\theta_{oa}) - \bar{u}_2(r_{aao}\sin\theta_{oa}) + \bar{u}_3 s_{oa}] \\
& + \hat{M}_{oa} e^{\tilde{u}_3\theta_{oa}} \hat{M}_{ao}^t (\bar{u}_1 x_a + \bar{u}_2 y_a + \bar{u}_3 z_a) \\
& = \hat{M}_{ob}[\bar{u}_1(r_{oob} - r_{bbo}\cos\theta_{ob}) - \bar{u}_2(r_{bbo}\sin\theta_{ob}) + \bar{u}_3 s_{ob}] \\
& + \hat{M}_{ob} e^{\tilde{u}_3\theta_{ob}} \hat{M}_{bo}^t (\bar{u}_1 x_b + \bar{u}_2 y_b + \bar{u}_3 z_b)
\end{aligned} \tag{4.30}$$

The details of the symbolic manipulations used in the preceding equations can be seen in the Appendix at the end of the paper.

C. Contact Equation for the Cams

Let $\vec{g}_a$ and $\vec{g}_b$ be the gradient vectors that belong to the cams L_a and L_b. Since the gradient vectors are normal to the surfaces of the cams, they must be aligned with each other when they are associated with the instantaneously coincident contact points Q_{ab} and Q_{ba}. This contact condition can be expressed by the following equation.

$$\vec{g}_a = -\lambda_{ab}\vec{g}_b \tag{4.31}$$

Here, λ_{ab} is defined as the *contact ratio* between the cams L_a and L_b. The minus sign is inserted into Eq. (4.31) because the cams are mostly convex and the gradient vectors are aligned oppositely at a convex-convex contact. Thus, $\lambda_{ab} > 0$ at a convex-convex contact and $\lambda_{ab} < 0$ at a convex-concave or concave-convex contact. If one of the cams happens to be flat (locally or globally), then the sign of λ_{ab} depends on the way the normal unit vector of the flat surface is oriented.

The surfaces of the cams can be described by the following equations, in which the scalar functions f_a and f_b are assumed to be differentiable twice with respect to their arguments.

$$f_a = f_a(\bar{\rho}_a) = f_a(x_a, y_a, z_a) = 0 \tag{4.32}$$

$$f_b = f_b(\bar{\rho}_b) = f_b(x_b, y_b, z_b) = 0 \tag{4.33}$$

The functions f_a and f_b lead to the following gradient expressions that represent the gradient vectors as column matrices respectively in the frames $\mathcal{F}_a$ and $\mathcal{F}_b$.

$$\bar{g}_a = \bar{g}_a^{(a)} = \bar{u}_1 \frac{\partial f_a}{\partial x_a} + \bar{u}_2 \frac{\partial f_a}{\partial y_a} + \bar{u}_3 \frac{\partial f_a}{\partial z_a} \tag{4.34}$$

$$\bar{g}_b = \bar{g}_b^{(b)} = \bar{u}_1 \frac{\partial f_b}{\partial x_b} + \bar{u}_2 \frac{\partial f_b}{\partial y_b} + \bar{u}_3 \frac{\partial f_b}{\partial z_b} \tag{4.35}$$

Thus, the matrix equivalent of Eq. (4.31) can be written as follows in the base frame $\mathcal{F}_o$.

$$\hat{C}^{(o,a)}\bar{g}_a = -\lambda_{ab}\hat{C}^{(o,b)}\bar{g}_b \tag{4.36}$$

As recalled from Part *A* of this section,

$$\hat{C}^{(o,b)} = \hat{C}^{(o,a)}\hat{C}^{(a,b)} = \hat{C}^{(o,a)} e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \tag{4.37}$$

Hence, Eq. (4.36) becomes

$$\bar{g}_a = -\lambda_{ab} e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \bar{g}_b \tag{4.38}$$

D. Summary of Equations and the Involved Variables

Here, the kinematic equations derived in Parts *A*, *B*, and *C* of this section are written again together as shown below.

$$\hat{M}_{oa} e^{\tilde{u}_3\theta_{oa}} \hat{M}_{ao}^t e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} = \hat{M}_{ob} e^{\tilde{u}_3\theta_{ob}} \hat{M}_{bo}^t \tag{4.39}$$

$$\begin{aligned}
& \hat{M}_{oa}[\bar{u}_1(r_{ooa} - r_{aao}\cos\theta_{oa}) - \bar{u}_2(r_{aao}\sin\theta_{oa}) + \bar{u}_3 s_{oa}] \\
& + \hat{M}_{oa} e^{\tilde{u}_3\theta_{oa}} \hat{M}_{ao}^t (\bar{u}_1 x_a + \bar{u}_2 y_a + \bar{u}_3 z_a) \\
& = \hat{M}_{ob}[\bar{u}_1(r_{oob} - r_{bbo}\cos\theta_{ob}) - \bar{u}_2(r_{bbo}\sin\theta_{ob}) + \bar{u}_3 s_{ob}] \\
& + \hat{M}_{ob} e^{\tilde{u}_3\theta_{ob}} \hat{M}_{bo}^t (\bar{u}_1 x_b + \bar{u}_2 y_b + \bar{u}_3 z_b)
\end{aligned} \tag{4.40}$$

$$f_a(x_a, y_a, z_a) = 0 \tag{4.41}$$

$$f_b(x_b, y_b, z_b) = 0 \tag{4.42}$$

$$\bar{g}_a = -\lambda_{ab} e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \bar{g}_b \tag{4.43}$$

The equations written above contain the following *fourteen* variables.

θ_{oa}, s_{oa}; θ_{ob}, s_{ob}; ϕ_{ab}, θ_{ab}, ψ_{ab};
x_a, y_a, z_a; x_b, y_b, z_b; λ_{ab}

The same equations contain the following *eleven* independent scalar equations.

Eq. (4.39) is a 3×3 orthonormal matrix equation. Therefore, it contains *three* independent scalar equations.

Eq'ns (4.40) and (4.43) are 3×1 column matrix equations. Therefore, they contain *six* independent scalar equations.

Eq'ns (4.41) and (4.42) are *two* independent scalar equations.

Since there are fourteen variables and eleven independent scalar equations, the *degree of freedom* of the system is

$$F = 3 \tag{4.44}$$

Therefore, if three of the variables are somehow specified, either as fixed or as input variables, the other eleven unspecified variables can be determined.

For the considered mechanism, depending on the preferred operational mode, one of θ_{oa} and s_{oa} is fixed and the other is specified as the input variable, while one of θ_{ob} and s_{ob} is fixed and the other is selected as the output variable, which is to be determined together with the other unspecified variables.

V. Semi-Analytical Solution of the Position Equations

To be more specific about the considered cam mechanism, let the cam surfaces be *ellipsoidal*. That is,

$$f_a = \xi_a^2 + \eta_a^2 + \zeta_a^2 - 1 = 0 \tag{5.1}$$

$$f_b = \xi_b^2 + \eta_b^2 + \zeta_b^2 - 1 = 0 \tag{5.2}$$

In Eq'ns (5.1) and (5.2), the six nondimensional variables (ξ_a to ζ_b) are defined as follows.

$$\{ \xi_a = x_a / d_{ax} ,\ \eta_a = y_a / d_{ay} ,\ \zeta_a = z_a / d_{az} \} \tag{5.3}$$

$$\{ \xi_b = x_b / d_{bx} ,\ \eta_b = y_b / d_{by} ,\ \zeta_b = z_b / d_{bz} \} \tag{5.4}$$

In Eq'ns (5.3) and (5.4), the six parameters d_{ax} to d_{bz} are the *semi-axis lengths* of the ellipsoidal surfaces.

Eq'ns (5.1) and (5.2) suggest to replace the distance-indicating variables (x_a to z_b) with the nondimensional variables (ξ_a to ζ_b) throughout the solution procedure. Thus, the same equations lead to the following matrix representations of the gradient vectors in the frames $\mathcal{F}_a$ and $\mathcal{F}_b$.

$$\bar{g}_a = \bar{u}_1\xi_a + \bar{u}_2\eta_a + \bar{u}_3\zeta_a \tag{5.5}$$

$$\bar{g}_b = \bar{u}_1\xi_b + \bar{u}_2\eta_b + \bar{u}_3\zeta_b \tag{5.6}$$

Note that, as suggested by Eq. (4.43), $\bar{g}_a$ and $\bar{g}_b$ are both simplified by the same factor (1/2) without loss of generality.

To start the solution, Eq. (4.39) can be written again as follows so as to relate ϕ_{ab} , θ_{ab} , and ψ_{ab} to θ_{oa} and θ_{ob} .

$$e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} = \hat{N}_1 e^{-\tilde{u}_3\theta_{oa}} \hat{N}_2 e^{\tilde{u}_3\theta_{ob}} \hat{N}_3 \tag{5.7}$$

In Eq. (5.7), the new constant matrices are defined so that

$$\hat{N}_1 = (\hat{M}_{ao}^t)^{-1} ,\ \hat{N}_2 = \hat{M}_{oa}^{-1}\hat{M}_{ob} ,\ \hat{N}_3 = \hat{M}_{bo}^t \tag{5.8}$$

Upon substituting Eq. (5.7), Eq. (4.43) becomes

$$\begin{aligned} &\bar{u}_1\xi_a + \bar{u}_2\eta_a + \bar{u}_3\zeta_a \\ &= -\lambda_{ab}\hat{N}_1 e^{-\tilde{u}_3\theta_{oa}} \hat{N}_2 e^{\tilde{u}_3\theta_{ob}} \hat{N}_3 (\bar{u}_1\xi_b + \bar{u}_2\eta_b + \bar{u}_3\zeta_b) \end{aligned} \tag{5.9}$$

Eq. (5.9) leads to the following equations for the variables ξ_a , η_a , and ζ_a .

$$\xi_a = -\lambda_{ab}\bar{u}_1^t \hat{N}_1 e^{-\tilde{u}_3\theta_{oa}} \hat{N}_2 e^{\tilde{u}_3\theta_{ob}} \hat{N}_3 (\bar{u}_1\xi_b + \bar{u}_2\eta_b + \bar{u}_3\zeta_b) \tag{5.10}$$

$$\eta_a = -\lambda_{ab}\bar{u}_2^t \hat{N}_1 e^{-\tilde{u}_3\theta_{oa}} \hat{N}_2 e^{\tilde{u}_3\theta_{ob}} \hat{N}_3 (\bar{u}_1\xi_b + \bar{u}_2\eta_b + \bar{u}_3\zeta_b) \tag{5.11}$$

$$\zeta_a = -\lambda_{ab}\bar{u}_3^t \hat{N}_1 e^{-\tilde{u}_3\theta_{oa}} \hat{N}_2 e^{\tilde{u}_3\theta_{ob}} \hat{N}_3 (\bar{u}_1\xi_b + \bar{u}_2\eta_b + \bar{u}_3\zeta_b) \tag{5.12}$$

Meanwhile, Eq. (4.40) can be written again as follows by inserting Eq'ns (5.3) and (5.4).

$$\begin{aligned} &\hat{M}_{oa}[\bar{u}_1(r_{ooa} - r_{aao}\cos\theta_{oa}) - \bar{u}_2(r_{aao}\sin\theta_{oa}) + \bar{u}_3 s_{oa}] \\ &+ \hat{M}_{oa} e^{\tilde{u}_3\theta_{oa}} \hat{M}_{ao}^t (\bar{u}_1 d_{ax}\xi_a + \bar{u}_2 d_{ay}\eta_a + \bar{u}_3 d_{az}\zeta_a) \\ &= \hat{M}_{ob}[\bar{u}_1(r_{oob} - r_{bbo}\cos\theta_{ob}) - \bar{u}_2(r_{bbo}\sin\theta_{ob}) + \bar{u}_3 s_{ob}] \\ &+ \hat{M}_{ob} e^{\tilde{u}_3\theta_{ob}} \hat{M}_{bo}^t (\bar{u}_1 d_{bx}\xi_b + \bar{u}_2 d_{by}\eta_b + \bar{u}_3 d_{bz}\zeta_b) \end{aligned} \tag{5.13}$$

When Eq'ns (5.10) to (5.12) are inserted, Eq. (5.13) can be arranged into the following linear equation for the variables ξ_b , η_b , and ζ_b .

$$\bar{u}_b\xi_b + \bar{v}_b\eta_b + \bar{w}_b\zeta_b = \bar{q}_b \tag{5.14}$$

In Eq. (5.14), the column matrices $\bar{u}_b$, $\bar{v}_b$, $\bar{w}_b$, and $\bar{q}_b$ are all known as functions of θ_{oa} , s_{oa} , θ_{ob} , s_{ob} , and λ_{ab} . Their expressions can be extracted from Eq. (5.13) without much difficulty after the substitution of Eq'ns (5.10) to (5.12).

The following equations can be derived from Eq. (5.14) in order to solve it for ξ_b , η_b , and ζ_b .

$$\left.\begin{aligned} \tilde{u}_b\bar{v}_b\eta_b + \tilde{u}_b\bar{w}_b\zeta_b &= \tilde{u}_b\bar{q}_b \\ \tilde{v}_b\bar{u}_b\xi_b + \tilde{v}_b\bar{w}_b\zeta_b &= \tilde{v}_b\bar{q}_b \\ \tilde{w}_b\bar{u}_b\xi_b + \tilde{w}_b\bar{v}_b\eta_b &= \tilde{w}_b\bar{q}_b \end{aligned}\right\} \tag{5.15}$$

Furthermore,

$$\left.\begin{aligned} (\bar{w}_b^t\tilde{u}_b\bar{v}_b)\eta_b &= \bar{w}_b^t\tilde{u}_b\bar{q}_b \\ (\bar{u}_b^t\tilde{v}_b\bar{w}_b)\zeta_b &= \bar{u}_b^t\tilde{v}_b\bar{q}_b \\ (\bar{v}_b^t\tilde{w}_b\bar{u}_b)\xi_b &= \bar{v}_b^t\tilde{w}_b\bar{q}_b \end{aligned}\right\} \tag{5.16}$$

It can be shown that

$$\bar{u}_b^t\tilde{v}_b\bar{w}_b = \bar{v}_b^t\tilde{w}_b\bar{u}_b = \bar{w}_b^t\tilde{u}_b\bar{v}_b = d_b \tag{5.17}$$

Here, d_b happens to be the determinant of the coefficient matrix $[\bar{u}_b\ \ \bar{v}_b\ \ \bar{w}_b]$ appearing implicitly in Eq. (5.14).

If $d_b \neq 0$, Eq. (5.16) gives ξ_b , η_b , and ζ_b as follows in terms of θ_{oa} , s_{oa} , θ_{ob} , s_{ob} , and λ_{ab} .

$$\xi_b = \bar{v}_b^t\tilde{w}_b\bar{q}_b / d_b \tag{5.18}$$

$$\eta_b = \bar{w}_b^t\tilde{u}_b\bar{q}_b / d_b \tag{5.19}$$

$$\zeta_b = \bar{u}_b^t\tilde{v}_b\bar{q}_b / d_b \tag{5.20}$$

When Eq'ns (5.18) to (5.20) are substituted, Eq'ns (5.10) to (5.12) give ξ_a , η_a , and ζ_a also as functions of θ_{oa} , s_{oa} , θ_{ob} , s_{ob} , and λ_{ab} .

For the considered mechanism, as mentioned before, one of θ_{oa} and s_{oa} is fixed and the other is specified as the input while one of θ_{ob} and s_{ob} is fixed. In other words, θ_{oa} and s_{oa} are known as well as one of θ_{ob} and s_{ob}. So, the remaining two variables λ_{ab} and the unknown one of θ_{ob} and s_{ob} can be found by solving Eq'ns (5.1) and (5.2) for them upon inserting the expressions of $\{\xi_a, \eta_a, \zeta_a\}$ and $\{\xi_b, \eta_b, \zeta_b\}$ derived above. The resulting two scalar equations can be denoted as follows.

$$f_a(\xi_a, \eta_a, \zeta_a) = F_a(\lambda_{ab}, q_{ob}) = 0 \tag{5.21}$$

$$f_b(\xi_b, \eta_b, \zeta_b) = F_b(\lambda_{ab}, q_{ob}) = 0 \tag{5.22}$$

In Eq'ns (5.21) and (5.22),

$$q_{ob} = \begin{cases} \theta_{ob} \text{ if } s_{ob} = d_{ob} = \text{constant} \\ s_{ob} \text{ if } \theta_{ob} = \delta_{ob} = \text{constant} \end{cases} \tag{5.23}$$

However, differently from the previous stages, the solution at this stage, i.e., the solution to Eq'ns (5.21) and (5.22) for λ_{ab} and q_{ob} cannot be obtained analytically because the expressions of the functions $F_a(\lambda_{ab}, q_{ob})$ and $F_b(\lambda_{ab}, q_{ob})$ are nonlinear and quite complicated. Therefore, a suitable numerical solution method must be used. Nevertheless, since there are only two equations to be solved numerically, the solution can still be obtained much more easily as compared to the difficulty of obtaining a completely numerical solution that involves eleven equations with eleven unknowns.

Actually, since the output is selected as q_{ob}, the solution procedure can be terminated as soon as its value is obtained.

However, if it is desired to find the values of the other variables as well, the following procedure can be pursued.

Once the values of λ_{ab} and q_{ob} are obtained as described above, they are used in Eq'ns (5.18) to (5.20) in order to find the values of ξ_b, η_b, and ζ_b. Afterwards, the values of ξ_b, η_b, and ζ_b are used in Eq'ns (5.10) to (5.12) in order to find the values of ξ_a, η_a, and ζ_a.

As for the angles ϕ_{ab}, θ_{ab}, and ψ_{ab}, they can be found from Eq. (5.7) by using the known values of θ_{oa} and θ_{ob}. For this purpose, Eq. (5.7) can be written as

$$e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} = \hat{B} \tag{5.24}$$

In Eq. (5.24), $\hat{B}$ is a known matrix, which is defined as

$$\hat{B} = \hat{N}_1 e^{-\tilde{u}_3\theta_{oa}} \hat{N}_2 e^{\tilde{u}_3\theta_{ob}} \hat{N}_3 \tag{5.25}$$

Eq. (5.24) can be used to find ϕ_{ab}, θ_{ab}, and ψ_{ab} by extracting the following equations out of it.

$$\bar{u}_3^t e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \bar{u}_3 = \bar{u}_3^t \hat{B} \bar{u}_3 = b_{33} \;\Rightarrow$$

$$\bar{u}_3^t e^{\tilde{u}_2\theta_{ab}} \bar{u}_3 = \bar{u}_3^t(\bar{u}_3 \cos\theta_{ab} + \bar{u}_1 \sin\theta_{ab}) = b_{33} \;\Rightarrow$$

$$\cos\theta_{ab} = b_{33} \tag{5.26}$$

Similarly,

$$\bar{u}_1^t e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \bar{u}_3 = \bar{u}_1^t e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} \bar{u}_3 = \bar{u}_1^t \hat{B} \bar{u}_3 = b_{13} \;\Rightarrow$$

$$\cos\phi_{ab} \sin\theta_{ab} = b_{13} \tag{5.27}$$

$$\bar{u}_2^t e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \bar{u}_3 = \bar{u}_2^t e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} \bar{u}_3 = \bar{u}_2^t \hat{B} \bar{u}_3 = b_{23} \;\Rightarrow$$

$$\sin\phi_{ab} \sin\theta_{ab} = b_{23} \tag{5.28}$$

$$\bar{u}_3^t e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \bar{u}_1 = \bar{u}_3^t e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \bar{u}_1 = \bar{u}_3^t \hat{B} \bar{u}_1 = b_{31} \;\Rightarrow$$

$$\sin\theta_{ab} \cos\psi_{ab} = -b_{31} \tag{5.29}$$

$$\bar{u}_3^t e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \bar{u}_2 = \bar{u}_3^t e^{\tilde{u}_2\theta_{ab}} e^{\tilde{u}_3\psi_{ab}} \bar{u}_2 = \bar{u}_3^t \hat{B} \bar{u}_2 = b_{32} \;\Rightarrow$$

$$\sin\theta_{ab} \sin\psi_{ab} = b_{32} \tag{5.30}$$

Eq. (5.26) implies that

$$\sin\theta_{ab} = \sigma d_{33} \tag{5.31}$$

In Eq. (5.31), σ is an arbitrary sign variable, i.e., $\sigma = \pm 1$, and

$$d_{33} = \sqrt{1 - b_{33}^2} \tag{5.32}$$

Hence, from Eq'ns (5.31) and (5.26), θ_{ab} is found as follows with a sign ambiguity represented by σ.

$$\theta_{ab} = \operatorname{atan}_2(\sigma d_{33}, b_{33}) \tag{5.33}$$

If $d_{33} = |\sin\theta_{ab}| \neq 0$, Eq. Pairs (5.27, 5.28) and (5.29, 5.30) give ϕ_{ab} and ψ_{ab} without any additional sign ambiguity as follows.

$$\phi_{ab} = \operatorname{atan}_2\left(\frac{b_{23}}{\sigma d_{33}}, \frac{b_{13}}{\sigma d_{33}}\right) = \operatorname{atan}_2(\sigma b_{23}, \sigma b_{13}) \tag{5.34}$$

$$\psi_{ab} = \operatorname{atan}_2\left(\frac{b_{32}}{\sigma d_{33}}, \frac{-b_{31}}{\sigma d_{33}}\right) = \operatorname{atan}_2(\sigma b_{32}, -\sigma b_{31}) \tag{5.35}$$

If $d_{33} = |\sin\theta_{ab}| = 0$, i.e., if $\theta_{ab} = 0$ or $\theta_{ab} = \pm\pi$, Eq. Pairs (5.27, 5.28) and (5.29, 5.30) cannot give ϕ_{ab} and ψ_{ab}. In such a case, although ϕ_{ab} and ψ_{ab} cannot be found separately, at least their combinations $\beta_{ab} = \phi_{ab} + \psi_{ab}$ and $\gamma_{ab} = \phi_{ab} - \psi_{ab}$ can still be found from Eq. (5.24) as shown below.

If $\theta_{ab} = 0$, Eq. (5.24) becomes

$$e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_2 0} e^{\tilde{u}_3\psi_{ab}} = e^{\tilde{u}_3\phi_{ab}} \hat{I} e^{\tilde{u}_3\psi_{ab}} = e^{\tilde{u}_3\phi_{ab}} e^{\tilde{u}_3\psi_{ab}} \;\Rightarrow$$

$$e^{\tilde{u}_3(\phi_{ab} + \psi_{ab})} = e^{\tilde{u}_3\beta_{ab}} = \hat{B} \tag{5.36}$$

If $\theta_{ab} = \pm\pi$, Eq. (5.24) becomes

$$e^{\tilde{u}_3\phi_{ab}} e^{\pm\tilde{u}_2\pi} e^{\tilde{u}_3\psi_{ab}} = e^{\tilde{u}_3\phi_{ab}} e^{-\tilde{u}_3\psi_{ab}} e^{\pm\tilde{u}_2\pi} = \hat{B} \;\Rightarrow$$

$$e^{\tilde{u}_3(\phi_{ab} - \psi_{ab})} = e^{\tilde{u}_3\gamma_{ab}} = \hat{B} e^{\mp\tilde{u}_2\pi} = \hat{B}^* \tag{5.37}$$

Then, Eq'ns (5.36) and (5.37) give β_{ab} and γ_{ab} as follows.

$$\beta_{ab} = \text{atan}_2(b_{21}, b_{11}) \tag{5.38}$$

$$\gamma_{ab} = \text{atan}_2(b_{21}^*, b_{11}^*) = \text{atan}_2(-b_{21}, -b_{11}) \tag{5.39}$$

VI. Velocity Equations

In this section, the orientation matrices are differentiated according to the following equations.

$$d(e^{\tilde{u}_k\theta}) / dt = \dot{\theta}\tilde{u}_k e^{\tilde{u}_k\theta} = \dot{\theta}e^{\tilde{u}_k\theta}\tilde{u}_k ;\ \ k = 1,2,3 \tag{6.1}$$

$$\dot{\hat{C}}^{(a,b)} = \tilde{\omega}_{b/a}^{(a)}\hat{C}^{(a,b)} \tag{6.2}$$

$$\dot{\hat{C}}^{(a,b)} = \hat{C}^{(a.b)}\tilde{\omega}_{b/a}^{(b)} \tag{6.3}$$

In Eq'ns (6.2) and (6.3),

$$\tilde{\omega}_{b/a}^{(a)} = \text{ssm}[\bar{\omega}_{b/a}^{(a)}] \ \text{ and } \ \tilde{\omega}_{b/a}^{(b)} = \text{ssm}[\bar{\omega}_{b/a}^{(b)}] \tag{6.4}$$

In Eq. Pair (6.4), $\bar{\omega}_{b/a}^{(a)}$ and $\bar{\omega}_{b/a}^{(b)}$ are the column matrix representations of the vector $\vec{\omega}_{b/a}$, which is the relative angular velocity of the frame $\mathcal{F}_b$ with respect to the frame $\mathcal{F}_a$.

Suppose $\hat{C}^{(a,b)}$ is obtained as a result of an *i-j-k* sequence of three successive rotations so that

$$\hat{C}^{(a,b)} = e^{\tilde{u}_i\phi}e^{\tilde{u}_j\theta}e^{\tilde{u}_k\psi} \tag{6.5}$$

Then, by using Eq'ns (6.2) and (6.3), it can be shown that

$$\bar{\omega}_{b/a}^{(a)} = \dot{\phi}\bar{u}_i + \dot{\theta}e^{\tilde{u}_i\phi}\bar{u}_j + \dot{\psi}e^{\tilde{u}_i\phi}e^{\tilde{u}_j\theta}\bar{u}_k \tag{6.6}$$

$$\bar{\omega}_{b/a}^{(b)} = \dot{\phi}e^{-\tilde{u}_k\psi}e^{-\tilde{u}_j\theta}\bar{u}_i + \dot{\theta}e^{-\tilde{u}_k\psi}\bar{u}_j + \dot{\psi}\bar{u}_k \tag{6.7}$$

The velocity equations associated with the considered cam mechanism are obtained as shown below by differentiating the relevant position equations that are used in the previous section.

Note that

$$e^{\tilde{u}_3\phi_{ab}}e^{\tilde{u}_2\theta_{ab}}e^{\tilde{u}_3\psi_{ab}} = \hat{C}^{(a,b)} \tag{6.8}$$

Hence, the differentiation of Eq. (5.7) leads to the following equation according to Eq. (6.6).

$$\begin{aligned}\bar{\omega}_{b/a}^{(a)} &= \dot{\phi}_{ab}\bar{u}_3 + \dot{\theta}_{ab}e^{\tilde{u}_3\phi_{ab}}\bar{u}_2 + \dot{\psi}_{ab}e^{\tilde{u}_3\phi_{ab}}e^{\tilde{u}_2\theta_{ab}}\bar{u}_3 \\ &= \dot{\theta}_{ob}\hat{N}_1 e^{-\tilde{u}_3\theta_{oa}}\hat{N}_2\bar{u}_3 - \dot{\theta}_{oa}\bar{u}_3\end{aligned} \tag{6.9}$$

The differentiation of Eq. (5.9) results in

$$\begin{aligned}&\bar{u}_1\dot{\xi}_a + \bar{u}_2\dot{\eta}_a + \bar{u}_3\dot{\zeta}_a \\ &= -\lambda_{ab}\hat{N}_1 e^{-\tilde{u}_3\theta_{oa}}\hat{N}_2 e^{\tilde{u}_3\theta_{ob}}\hat{N}_3(\bar{u}_1\dot{\xi}_b + \bar{u}_2\dot{\eta}_b + \bar{u}_3\dot{\zeta}_b) \\ &\quad -\dot{\lambda}_{ab}\hat{N}_1 e^{-\tilde{u}_3\theta_{oa}}\hat{N}_2 e^{\tilde{u}_3\theta_{ob}}\hat{N}_3(\bar{u}_1\xi_b + \bar{u}_2\eta_b + \bar{u}_3\zeta_b) \\ &\quad +\lambda_{ab}\dot{\theta}_{oa}\hat{N}_1 e^{-\tilde{u}_3\theta_{oa}}\tilde{u}_3\hat{N}_2 e^{\tilde{u}_3\theta_{ob}}\hat{N}_3(\bar{u}_1\xi_b + \bar{u}_2\eta_b + \bar{u}_3\zeta_b) \\ &\quad -\lambda_{ab}\dot{\theta}_{ob}\hat{N}_1 e^{-\tilde{u}_3\theta_{oa}}\hat{N}_2 e^{\tilde{u}_3\theta_{ob}}\tilde{u}_3\hat{N}_3(\bar{u}_1\xi_b + \bar{u}_2\eta_b + \bar{u}_3\zeta_b)\end{aligned} \tag{6.10}$$

The following equations are obtained upon the differentiation of Eq'ns (5.1), (5.2), and (5.13).

$$\xi_a\dot{\xi}_a + \eta_a\dot{\eta}_a + \zeta_a\dot{\zeta}_a = 0 \tag{6.11}$$

$$\xi_b\dot{\xi}_b + \eta_b\dot{\eta}_b + \zeta_b\dot{\zeta}_b = 0 \tag{6.12}$$

$$\begin{aligned}&\hat{M}_{oa}[(\bar{u}_1\sin\theta_{oa} - \bar{u}_2\cos\theta_{oa})r_{aao}\dot{\theta}_{oa} + \bar{u}_3\dot{s}_{oa}] \\ &+\dot{\theta}_{oa}\hat{M}_{oa}e^{\tilde{u}_3\theta_{oa}}\tilde{u}_3\hat{M}_{ao}^t(\bar{u}_1 d_{ax}\xi_a + \bar{u}_2 d_{ay}\eta_a + \bar{u}_3 d_{az}\zeta_a) \\ &+\hat{M}_{oa}e^{\tilde{u}_3\theta_{oa}}\hat{M}_{ao}^t(\bar{u}_1 d_{ax}\dot{\xi}_a + \bar{u}_2 d_{ay}\dot{\eta}_a + \bar{u}_3 d_{az}\dot{\zeta}_a) \\ &= \hat{M}_{ob}[(\bar{u}_1\sin\theta_{ob} - \bar{u}_2\cos\theta_{ob})r_{bbo}\dot{\theta}_{ob} + \bar{u}_3\dot{s}_{ob}] \\ &+\dot{\theta}_{ob}\hat{M}_{ob}e^{\tilde{u}_3\theta_{ob}}\tilde{u}_3\hat{M}_{bo}^t(\bar{u}_1 d_{bx}\xi_b + \bar{u}_2 d_{by}\eta_b + \bar{u}_3 d_{bz}\zeta_b) \\ &+\hat{M}_{ob}e^{\tilde{u}_3\theta_{ob}}\hat{M}_{bo}^t(\bar{u}_1 d_{bx}\dot{\xi}_b + \bar{u}_2 d_{by}\dot{\eta}_b + \bar{u}_3 d_{bz}\dot{\zeta}_b)\end{aligned} \tag{6.13}$$

VII. Solution of the Velocity Equations

The solution can be started from Eq. (6.10), which gives the rates $\dot{\xi}_a$, $\dot{\eta}_a$, and $\dot{\zeta}_a$ with the following expressions.

$$\dot{\xi}_a = h_{11}\dot{\xi}_b + h_{12}\dot{\eta}_b + h_{13}\dot{\zeta}_b + h_{14}\dot{\lambda}_{ab} + h_{15}\dot{\theta}_{ob} + h_{16}\dot{\theta}_{oa} \tag{6.14}$$

$$\dot{\eta}_a = h_{21}\dot{\xi}_b + h_{22}\dot{\eta}_b + h_{23}\dot{\zeta}_b + h_{24}\dot{\lambda}_{ab} + h_{25}\dot{\theta}_{ob} + h_{26}\dot{\theta}_{oa} \tag{6.15}$$

$$\dot{\zeta}_a = h_{31}\dot{\xi}_b + h_{32}\dot{\eta}_b + h_{33}\dot{\zeta}_b + h_{34}\dot{\lambda}_{ab} + h_{35}\dot{\theta}_{ob} + h_{36}\dot{\theta}_{oa} \tag{6.16}$$

When Eq'ns (6.14) to (6.16) are substituted into Eq. (6.13), the following linear equation is obtained for the rates $\dot{\xi}_b$, $\dot{\eta}_b$, and $\dot{\zeta}_b$.

$$\bar{\mu}_1\dot{\xi}_b + \bar{\mu}_2\dot{\eta}_b + \bar{\mu}_3\dot{\zeta}_b = \bar{\mu}_0 \tag{6.17}$$

In Eq. (6.17), $\bar{\mu}_0$ is a linear function of $\dot{\lambda}_{ab}$, $\dot{\theta}_{ob}$, $\dot{s}_{ob}$, $\dot{\theta}_{oa}$, and $\dot{s}_{oa}$. That is,

$$\bar{\mu}_0 = \bar{n}_1\dot{\lambda}_{ab} + \bar{n}_2\dot{\theta}_{ob} + \bar{n}_3\dot{s}_{ob} + \bar{n}_4\dot{\theta}_{oa} + \bar{n}_5\dot{s}_{oa} \tag{6.18}$$

In Eq'ns (6.17) and (6.18), the coefficients of the rates are known as functions of the currently available position of the mechanism.

Eq. (6.17) can be solved similarly as Eq. (5.14) is solved. Thus, its solution can be expressed as follows if $d_r \neq 0$.

$$\dot{\xi}_b = \bar{\mu}_2^t\tilde{\mu}_3\bar{\mu}_0 / d_r \tag{6.19}$$

$$\dot{\eta}_b = \bar{\mu}_3^t\tilde{\mu}_1\bar{\mu}_0 / d_r \tag{6.20}$$

$$\dot{\zeta}_b = \bar{\mu}_1^t\tilde{\mu}_2\bar{\mu}_0 / d_r \tag{6.21}$$

The determinant d_r is also defined similarly as in Eq. (5.17). In other words,

$$d_r = \bar{\mu}_1^t\tilde{\mu}_2\bar{\mu}_3 = \bar{\mu}_2^t\tilde{\mu}_3\bar{\mu}_1 = \bar{\mu}_3^t\tilde{\mu}_1\bar{\mu}_2 \tag{6.22}$$

By means of Eq'ns (6.14) to (6.21), the rates $\dot{\xi}_a$ to $\dot{\zeta}_b$ become available as linear functions of $\dot{\lambda}_{ab}$, $\dot{\theta}_{ob}$, $\dot{s}_{ob}$, $\dot{\theta}_{oa}$, and $\dot{s}_{oa}$. Therefore, when they are inserted into Eq'ns (6.11) and (6.12), the following linear equations are obtained.

$$k_{11}\dot{\lambda}_{ab} + k_{12}\dot{\theta}_{ob} + k_{13}\dot{s}_{ob} + k_{14}\dot{\theta}_{oa} + k_{15}\dot{s}_{oa} = 0 \quad (6.23)$$

$$k_{21}\dot{\lambda}_{ab} + k_{22}\dot{\theta}_{ob} + k_{23}\dot{s}_{ob} + k_{24}\dot{\theta}_{oa} + k_{25}\dot{s}_{oa} = 0 \quad (6.24)$$

Upon the elimination of $\dot{\lambda}_{ab}$, Eq'ns (6.23) and (6.24) merge into the following single equation.

$$k_{32}\dot{\theta}_{ob} + k_{33}\dot{s}_{ob} + k_{34}\dot{\theta}_{oa} + k_{35}\dot{s}_{oa} = 0 \quad (6.25)$$

If the output is selected to be θ_{ob}, then $\dot{s}_{ob} = 0$ and the value of $\dot{\theta}_{ob}$ is found as

$$\dot{\theta}_{ob} = -(k_{34}\dot{\theta}_{oa} + k_{35}\dot{s}_{oa}) / k_{32} \quad (6.26)$$

If the output is selected to be s_{ob}, then $\dot{\theta}_{ob} = 0$ and the value of $\dot{s}_{ob}$ is found as

$$\dot{s}_{ob} = -(k_{34}\dot{\theta}_{oa} + k_{35}\dot{s}_{oa}) / k_{33} \quad (6.27)$$

Afterwards, if desired, the values of $\dot{\xi}_a$ to $\dot{\zeta}_b$ can be found by using Eq'ns (6.19) to (6.21) and Eqn's (6.14) to (6.16).

Also if desired, Eq. (6.9) can be used to find the values of the remaining rates $\dot{\phi}_{ab}$, $\dot{\theta}_{ab}$, and $\dot{\psi}_{ab}$. For this purpose, Eq. (6.9) can be manipulated as follows.

$$\dot{\phi}_{ab}\bar{u}_3 + \dot{\theta}_{ab}\bar{u}_2 + \dot{\psi}_{ab}e^{\tilde{u}_2\theta_{ab}}\bar{u}_3 = \bar{\omega}_{ab} \quad \Rightarrow$$

$$\bar{u}_1(\dot{\psi}_{ab}\sin\theta_{ab}) + \bar{u}_2\dot{\theta}_{ab} + \bar{u}_3(\dot{\phi}_{ab} + \dot{\psi}_{ab}\cos\theta_{ab}) = \bar{\omega}_{ab} \quad (6.28)$$

In Eq. (6.28), $\bar{\omega}_{ab}$ is available as

$$\bar{\omega}_{ab} = e^{-\tilde{u}_3\phi_{ab}}(\dot{\theta}_{ob}\hat{N}_1 e^{-\tilde{u}_3\theta_{oa}}\hat{N}_2\bar{u}_3 - \dot{\theta}_{oa}\bar{u}_3) \quad (6.29)$$

Eq. (6.28) leads to the following scalar equations.

$$\dot{\psi}_{ab}\sin\theta_{ab} = \bar{u}_1^t\bar{\omega}_{ab} = \omega'_{ab} \quad (6.30)$$

$$\dot{\theta}_{ab} = \bar{u}_2^t\bar{\omega}_{ab} = \omega''_{ab} \quad (6.31)$$

$$\dot{\phi}_{ab} + \dot{\psi}_{ab}\cos\theta_{ab} = \bar{u}_3^t\bar{\omega}_{ab} = \omega'''_{ab} \quad (6.32)$$

Eq. (6.31) has already given the value of $\dot{\theta}_{ab}$.

If $\sin\theta_{ab} \neq 0$, Eq'ns (6.30) and (6.32) give the values of $\dot{\psi}_{ab}$ and $\dot{\phi}_{ab}$ as follows.

$$\dot{\psi}_{ab} = \omega'_{ab} / \sin\theta_{ab} \quad (6.33)$$

$$\dot{\phi}_{ab} = \omega'''_{ab} - \omega'_{ab}\,\mathrm{cotan}\,\theta_{ab} \quad (6.34)$$

If $\sin\theta_{ab} = 0$, i.e., if $\theta_{ab} = 0$ or $\theta_{ab} = \pm\pi$, $\dot{\psi}_{ab}$ and $\dot{\phi}_{ab}$ cannot be found separately. This is because Eq. (6.30) becomes trivial as $0 = 0$ and there remains only Eq. (6.32), which gives the following combination of $\dot{\psi}_{ab}$ and $\dot{\phi}_{ab}$.

$$\dot{\phi}_{ab} + \sigma'\dot{\psi}_{ab} = \omega'''_{ab} \quad (6.35)$$

In Eq. (6.35), $\sigma' = 1$ if $\theta_{ab} = 0$ and $\sigma' = -1$ if $\theta_{ab} = \pm\pi$.

VIII. Conclusion

This paper presents a special notation and methodology that can be used conveniently for the kinematic analysis of the mechanisms that involve rolling kinematic pairs such as cams.

The convenience has been demonstrated on a three-link spatial cam mechanism. Indeed, it has been possible to write the loop closure and cam contact equations systematically in such a way that they are suitable for further symbolic manipulations.

As a further manipulation, the position-level equations have been solved for the unspecified variables in a semi-analytical way so that the number of equations that necessitate a numerical solution is reduced from eleven to two.

As another further manipulation, the position-level equations have been differentiated in order to obtain the velocity-level equations, which also turn out to be suitable for symbolic manipulations. Thus, it has been possible to solve them analytically for the rates of the unspecified variables.

Another conveniece of the methodology is the introduction of the contact ratio between the gradient vectors as an auxiliary variable. It has been demonstrated that this variable does indeed facilitate writing and solving the kinematic equations.

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Appendix

The rotation matrices formed by the basic column matrices have the following properties that facilitate symbolic manipulations. These properties are excerpted from [12].

- $\det(e^{\tilde{u}_k\theta}) = 1$
- $(e^{\tilde{u}_k\theta})^{-1} = (e^{\tilde{u}_k\theta})^t = e^{-\tilde{u}_k\theta}$
- $e^{\tilde{u}_k\theta} e^{\tilde{u}_k\phi} = e^{\tilde{u}_k(\theta+\phi)}$
- $e^{\tilde{u}_i\theta} e^{\tilde{u}_k\phi} \neq e^{\tilde{u}_k\phi} e^{\tilde{u}_i\theta} \neq e^{\tilde{u}_i\theta+\tilde{u}_k\phi}$ if $i \neq k$
- $e^{\tilde{u}_k\theta}\bar{u}_k = \bar{u}_k$ and $\bar{u}_k^t e^{\tilde{u}_k\theta} = \bar{u}_k^t$
- $e^{\tilde{u}_i\theta}\bar{u}_j = \bar{u}_j \cos\theta + \sigma_{ijk}\bar{u}_k \sin\theta$
- $\bar{u}_j^t e^{\tilde{u}_i\theta} = \bar{u}_j^t \cos\theta + \sigma_{jik}\bar{u}_k^t \sin\theta$
- $e^{\tilde{u}_i\pi/2} e^{\tilde{u}_j\theta} = e^{\sigma_{ijk}\tilde{u}_k\theta} e^{\tilde{u}_i\pi/2}$

  $$\sigma_{ijk} = \begin{cases} +1 & \text{if } ijk = 123,\ 231,\ 312 \\ -1 & \text{if } ijk = 321,\ 132,\ 213 \end{cases}$$
- $\mathrm{ssm}(e^{\tilde{u}_i\theta}\bar{u}_k) = e^{\tilde{u}_i\theta}\tilde{u}_k e^{-\tilde{u}_i\theta}$
- $d(e^{\tilde{u}_k\theta}) / d\theta = e^{\tilde{u}_k\theta}\tilde{u}_k = \tilde{u}_k e^{\tilde{u}_k\theta}$
- $e^{\tilde{u}_i\pi} e^{\tilde{u}_i\theta} = e^{\tilde{u}_i\theta} e^{\tilde{u}_i\pi}$
- $e^{\tilde{u}_i\pi} e^{\tilde{u}_k\theta} = e^{-\tilde{u}_k\theta} e^{\tilde{u}_i\pi}$ if $i \neq k$